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Sustained positive consumption in a model of stochastic growth: The role of risk aversion ☆

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Abstract

In a stochastic economy, long run consumption and output may not be bounded away from zero even when productivity is arbitrarily high near zero and uncertainty is arbitrarily small. In the one-sector stochastic optimal growth model with i.i.d. production shocks, we characterize the nature of preferences that lead to this phenomenon for a stochastic Cobb–Douglas technology. For the general version of the model, we outline sufficient conditions under which the economy expands its capital stock near zero and long run consumption is bounded away from zero with certainty. Our conditions highlight the important role played by risk aversion for small consumption levels.

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1. Introduction

An important concern in the economic theory of growth is whether positive levels of consumption are sustained in the long run. The latter may be viewed as a minimal condition for the long run survival of the economic system. It is this concern that explains the focus of economic

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growth theory on convergence to an "interior" steady state. In models of growth where there is no uncertainty affecting the return on investment, whether or not long run consumption is positive is determined by a comparison of the marginal productivity (or rate of return on investment) near zero and the degree of impatience to consume (as reflected in the discount rate); these factors affect the incentive of agents to withhold current consumption and expand capital when capital stocks are sufficiently small. However, there are additional complications when productivity or the return on investment is uncertain, so that future consumption is essentially a risky prospect compared to the relative certainty of current consumption. In that case, consumption utility, and in particular, the nature of risk preference of economic agents, is likely to play an important role in determining whether or not the economy invests sufficiently to expand when output and capital stocks are small. This paper is an attempt to understand the nature of intertemporal preferences and technology that determine whether or not a dynamic stochastic economy sustains positive consumption in the long run with probability one.

Our analysis is carried out in the framework of the well-known one sector model of optimal stochastic growth with a strictly concave "bounded growth" production function and a strictly concave utility function [4]. In the *deterministic* version of this model,[1] positive consumption and capital are sustained in the long run, and the economy expands near zero, if, and only if, the net marginal productivity at zero exceeds the discount rate. Indeed, under the latter condition, there is a unique non-zero optimal steady state (the modified golden rule), and from every strictly positive initial stock the economy converges to this steady state. In particular, if the marginal productivity at zero is infinite, then long run consumption and capital are bounded away from zero no matter how heavily the future is discounted.

The situation is qualitatively different when the production technology is subject to random shocks over time. This is demonstrated in a striking example by Mirman and Zilcha [11]; they show that even if the technology is infinitely productive at zero with probability one, and even if the extent of discounting is arbitrarily mild, optimal capital and consumption may be arbitrarily close to zero in the long run. More specifically, they consider a strictly concave Cobb–Douglas production technology with multiplicative random shocks that are independent and uniformly distributed. For this technology, even the "worst" possible realization of the production function exhibits infinite productivity at zero. The support of the distribution of shocks (the extent of uncertainty) is allowed to be arbitrarily small. They show that given any discount factor less than one, and the parameters of the specific stochastic technology, *there exists* a smooth, strictly concave "regular" utility function such that in the dynamic economy (with this utility function, the specified stochastic technology and discount factor), capital and consumption fall below *any* strictly positive threshold infinitely often with probability one,[2] and in particular, there does not exist any invariant distribution (stochastic steady state) whose support is bounded away from zero. This is particularly striking when we consider the fact that if the distribution of shocks is degenerate, this economy necessarily expands from sufficiently small stocks and the consumption level on every optimal path is bounded away from zero. This brings out a fundamental qualitative difference between deterministic and stochastic models of growth.[3]

[1] In this discussion and the rest of this paper, assume full depreciation of capital every period.

[2] See also [14].

[3] The phenomenon described in [11] is qualitatively different from that outlined by Kamihigashi [7] who shows that if the marginal product at zero is *finite*, then every feasible path (including, therefore, any optimal path) converges *almost surely* to zero, provided the random shocks are "sufficiently volatile". The latter result is not driven by any property of the utility function.

The Mirman–Zilcha example illustrates a possibility that clearly deserves recognition in any study focusing on sustaining positive consumption under uncertainty and the existence of a stochastic steady state whose support is bounded away from zero. The fact that this problem does not arise for certain specific functional forms used widely in macroeconomics (such as the combination of the constant relative risk aversion utility function and the Cobb–Douglas production function) does not actually reduce its importance; indeed, it raises questions about the extent to which some of the qualitative results on long run behavior generated by using these specific functional forms are robust to alternative specifications of preferences and technology. More generally, we need to understand *the extent of the problem*, i.e., how pervasive is the phenomenon indicated by the Mirman–Zilcha example within the space of admissible economies? Moreover, one would like to understand the economic forces that lead to this phenomenon. The existing literature sheds no light on these issues. Though Mirman and Zilcha [11] show the existence of a utility function (for each value of the discount factor and the specific technology) that leads to the indicated outcome, they do not explicitly specify the utility function or qualitatively characterize the class of utility functions that can give rise to it.

The first objective of this paper is to understand the specific properties of the consumption preferences that lead to the phenomenon described in the Mirman–Zilcha example. We choose the same stochastic technology used in their example, i.e., a Cobb–Douglas production function with multiplicative and uniformly distributed shock, and outline verifiable sufficient conditions on the utility function under which capital and consumption are arbitrarily close to zero infinitely often with probability one. Under a mild condition, we derive a tight *necessary and sufficient condition* for this phenomenon to occur when discounting is sufficiently strong. The condition requires that as consumption goes to zero, the Arrow–Pratt measure of relative risk aversion diverges to infinity at a sufficiently fast rate. This indicates that the source of the problem identified in the Mirman–Zilcha example lies in risk aversion and, in particular, in the manner in which risk aversion explodes near zero. We illustrate this condition using a specific family of "exponential-power" utility functions. We then consider the situation where discounting is arbitrarily mild, and explicitly specify a utility function from the "exponential-power" family for which the phenomenon occurs.

We also indicate that if productivity is *bounded*, then the phenomenon can occur even if risk aversion is bounded, for instance with a constant relative risk aversion utility function; in particular, no matter how high the productivity at zero, the economy may not be bounded away from zero if (the constant relative) risk aversion is large enough. This also suggests that there are no "safe" utility functions. The nature of the technology determines the class of utility functions for which the economy is not bounded away from zero, and the latter may include some of the widely used utility functions.

The characterization above naturally leads to the question of identifying *general* verifiable conditions on preferences and technology under which the problem does *not* arise. The existing literature has identified some strong assumptions that ensure this.[4] Brock and Mirman [4] and Mirman and Zilcha [10] impose two conditions that ensure expansion of capital and consumption near zero even under the worst realization of the stochastic technology. They require marginal productivity at zero to be infinite for all realizations of the random shock, and impose a strictly

[4] A number of papers directly impose conditions on endogenously determined elements such as the optimal policy function, or the stochastic process of optimal capital stocks. See, among others, [3,9].

positive probability mass on the "worst" realization of the technology.[5] These are clearly strong assumptions. For one thing, they do not allow for random shocks with continuous distributions. Further, the requirement that the worst possible realization of the production technology is infinitely productive at zero rules out economies where productivity may not be high under bad realizations of the technology shock. In a more recent contribution, Chatterjee and Shukayev [6] weaken the productivity requirement above; they require the lowest possible (net) marginal productivity at zero to exceed the discount rate; they show that if, in addition, the utility function is bounded below, then the economy is bounded away from zero with probability one (though the economy may not necessarily grow near zero and the lower bound on long run consumption may depend on the initial condition). The restriction that the utility function is bounded below rules out a large class of utility functions that are widely used in the macroeconomic growth literature.[6] Also, in many situations the economy may not be sufficiently productive relative to the discount rate for bad realizations of the production shock; unless such states occur with high probability, one may still expect the economy to be bounded away from zero as long as they are balanced by high productivity in better states of the world. What should matter is the behavior of some kind of suitably modified version of "average" or expected productivity rather than the productivity in the worst case scenario. Finally, the literature on optimal harvesting of renewable resources under uncertainty contains some conditions under which resource stocks are bounded away from extinction. In a model that allows for non-concave production functions, Mitra and Roy [13] outline a *joint restriction* on preferences and the production function that ensures growth of stocks near zero.[7] However, their analysis does not shed any light on the restriction such a condition implies on the class of utility functions (for any given production function and discount factor).

We consider the *general version* of the one sector stochastic growth model and provide a set of verifiable sufficient conditions under which the economy sustains positive consumption in the long run. These conditions are also useful for practitioners who work with stochastic growth models and need to make sure that their specification of preferences and technology generates a stochastic steady state whose support is bounded away from zero. In particular, we provide conditions under which the optimal policy function exhibits "growth near zero" under the worst realization of the production function so that independent of initial capital, long run capital and consumption are *uniformly* bounded below by a positive number. We also provide conditions for a weaker form of avoidance of zero where capital and consumption are bounded away from zero with probability one (though the bound may depend on the initial condition). Our general theoretical conditions are in the nature of restrictions on the (limiting) behavior at zero of the *expected* marginal productivity modified by a factor that involves the ratio of marginal utilities of consumptions (that provides a verifiable bound on the marginal rate of substitution between current consumption and future stochastic consumption). For any given technology and discount factor, the behavior of this ratio of marginal utilities near zero is the key restriction needed on the class of utility functions in order to ensure sustained positive consumption. We show that the behavior of this ratio is closely related to the degree of (Arrow–Pratt) relative risk aversion, and provide a condition that involves explicit restrictions on the degree of risk aversion at

[5] This is the production function corresponding to the lower bound of the support of the random shock if the production functions are ordered by realizations of the shock.

[6] Examples include $u(c) = \ln c$, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 1$, where $u(0) = -\infty$.

[7] Olson and Roy [17] provide a similar condition for avoidance of extinction of a renewable resource in a stochastic non-convex model where the utility depends on resource consumption as well as the resource stock.

zero. Higher the risk aversion at zero, higher the discounted expected marginal productivity at zero that is needed to ensure sustained positive consumption. If the expected marginal productivity at zero is infinite, sustained positive consumption is ensured as long as risk aversion is bounded.

For utility functions that are bounded below, we also provide sufficient conditions for consumption to be bounded away from zero using the first elasticity of the utility function near zero. Unlike some of the existing conditions, our conditions allow for the possibility that for bad realizations of the random shock, the net marginal productivity at zero may be lower than the discount rate. We show that if utility is bounded below, under some mild restrictions on the production function, infinite *expected* marginal productivity at zero is sufficient for sustained positive consumption no matter how small the discount factor.

The rest of the paper is organized as follows. Section 2 discusses the model and some preliminary results. Section 3 discusses the problem of the economy being nowhere bounded away from zero and provides necessary and sufficient conditions for this phenomenon for a specific Cobb–Douglas stochastic technology. Section 4 contains general theoretical results providing sufficient conditions for the economy to exhibit growth near zero which ensures a uniform positive lower bound on long run consumption independent of initial condition. Section 5 discusses sufficient conditions for a slightly weaker property that also ensures sustained positive consumption in the long run.

2. Model

We consider an infinite horizon one-good representative agent economy. Time is discrete and is indexed by $t = 0, 1, 2, \dots$. The initial stock of output $y_0 > 0$ is given. At each date $t \geqslant 0$, the representative agent observes the current stock of output $y_t \in \mathbb{R}_+$ and chooses the level of current investment x_t, and the current consumption level c_t, such that

$$c_t \geqslant 0, \qquad x_t \geqslant 0, \qquad c_t + x_t \leqslant y_t.$$

This generates y_{t+1}, the output next period, through the relation

$$y_{t+1} = f(x_t, r_{t+1})$$

where $f(.,.)$ is the "aggregate" production function and r_{t+1} is a random production shock realized at the beginning of period $(t+1)$.

The following assumption is made on the sequence of random shocks:

(A.1) *$\{r_t\}_{t=1}^{\infty}$ is an independent and identically distributed random process defined on a probability space $(\Omega, \mathcal{F}, P)$, where the marginal distribution function is denoted by F. The support of this distribution is a compact set $A \subset \mathbb{R}$.*

The production function $f : \mathbb{R}_+ \times A \to \mathbb{R}_+$ is assumed to satisfy the following:

(T.1) *For all $r \in A$, $f(x, r)$ is concave in x on $\mathbb{R}_+$.*

(T.2) *For all $r \in A$, $f(0, r) = 0$.*

(T.3) *$f(x, r)$ is continuous in (x, r) on $\mathbb{R}_+ \times A$. For each $r \in A$, $f(x, r)$ is differentiable in x on $\mathbb{R}_{++}$ and, further, $f'(x, r) = \frac{\partial f(x,r)}{\partial x} > 0$ on $\mathbb{R}_{++} \times A$.*

Assumptions (T.1)–(T.3) are standard monotonicity, concavity and smoothness restrictions on production. For any investment level $x \geqslant 0$, let the upper and lower bound of the support of output next period be denoted by $\overline{f}(x)$ and $\underline{f}(x)$, respectively. In particular,

$$\overline{f}(x) = \max_{r \in A} f(x, r), \qquad \underline{f}(x) = \min_{r \in A} f(x, r). \tag{1}$$

It is easy to check that $\underline{f}(x)$ is continuous, concave and strictly increasing on $\mathbb{R}_+$. Further, $\overline{f}(x)$ is continuous and strictly increasing on $\mathbb{R}_+$. We assume that:

(T.4) $\limsup_{x \to 0}[\frac{\overline{f}(x)}{\underline{f}(x)}] < \infty$.

Assumption (T.4) imposes a bound on the extent of fluctuation in output that can be caused by the random shock. Note that (T.4) is always satisfied when the production shock is multiplicative, i.e., $f(x, r) = rh(x)$ as long as A is a compact subset of $\mathbb{R}_{++}$.

For each $r \in A$, let $D_+ f(0, r)$ denote the "limiting" marginal product at zero investment:

$$D_+ f(0, r) = \lim_{x \downarrow 0} f'(x, r).$$

Using the concavity of $f(x, r)$ in x and $f(0, r) = 0$,

$$D_+ f(0, r) = \lim_{x \downarrow 0} \frac{f(x, r)}{x}. \tag{2}$$

Let

$$\nu = \lim_{x \downarrow 0} \frac{\underline{f}(x)}{x}. \tag{3}$$

Then, $\nu = +\infty$ if the production function satisfies the well-known Uzawa–Inada condition at zero. We assume that

(T.5) $\nu > 1$ *and* $\lim_{x \to \infty} \sup \frac{\overline{f}(x)}{x} < 1$.

The first part of assumption (T.5) ensures that it is feasible for capital and output to grow with certainty in a neighborhood of zero. The second part of the assumption implies that the technology exhibits bounded growth.

We denote by u the one period utility function from consumption. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The utility function $u : \mathbb{R}_+ \to \overline{\mathbb{R}}$ satisfies the following restrictions:

(U.1) *u is strictly increasing, continuous and strictly concave on $\mathbb{R}_+$ (on $\mathbb{R}_{++}$ if $u(0) = -\infty$); $u(c) \to u(0)$ as $c \to 0$.*
(U.2) *u is twice continuously differentiable on $\mathbb{R}_{++}$; $u'(c) > 0$, $u''(c) < 0$, $\forall c > 0$.*
(U.3) $\lim_{c \to 0} u'(c) = +\infty$.

Assumptions (U.1) and (U.2) are standard. Note that we allow the utility of zero consumption to be $-\infty$. (U.3) requires that the utility function satisfy the Uzawa–Inada condition at zero.

Let $\delta \in (0,1)$ denote the time discount factor. Given the initial stock $y_0 > 0$, the representative agent's objective is to maximize the expected value of the discounted sum of utilities from consumption:

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right].$$

We follow the standard definitions of policy, Markovian policy, stationary policy, optimal policy and value function as used in the literature on stochastic dynamic programming. Let $V(y)$ denote the *value function* defined on $\mathbb{R}_{++}$. Under assumption (T.5), it is easy to check that

$$-\infty < V(y) < +\infty, \quad \forall y > 0.$$

Standard dynamic programming arguments imply that there exists a unique optimal policy, that this policy is stationary, and that the value function satisfies the functional equation:

$$V(y) = \max_{0 \leqslant x \leqslant y} \bigl[u(y-x) + \delta E\bigl[V\bigl(f(x,r)\bigr)\bigr]\bigr]. \tag{4}$$

It can be shown that $V(y)$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$. Further, the maximization problem on the right-hand side of (4) has a unique solution, denoted by $x(y)$. The stationary policy generated by the function $x(y)$ is the optimal policy and we refer to $x(y)$ as the optimal investment function; $c(y) = y - x(y)$ is the optimal consumption function.

Given initial stock $y > 0$, the stochastic process of optimal output $\{y_t(y,\omega)\}$ evolves over time according to the transition rule:

$$y_t(y,\omega) = f\bigl(x\bigl(y_{t-1}(y,\omega)\bigr), \omega_t\bigr) \quad \text{for } t \geqslant 1 \tag{5}$$

and $y_0(y,\omega) = y$.

Using standard arguments in the literature, (U.3) can be used to show that:

Lemma 1. *For all $y > 0$, $x(y) > 0$ and $c(y) > 0$.*

Lemma 2. *$x(y)$ and $c(y)$ are continuous and strictly increasing in y on $\mathbb{R}_+$.*

Lemmas 1 and 2 imply that consumption is bounded away from zero along any realized path of the economy if, and only if, capital and output are bounded away from zero. Further, Lemma 2 implies that between any two periods, consumption expands if, and only if, capital and output expand. Next, we note that the stochastic Ramsey–Euler equation holds:

Lemma 3. *For all $y > 0$,*

$$u'\bigl(c(y)\bigr) = \delta E\bigl[u'\bigl(c\bigl(f\bigl(x(y),r\bigr)\bigr)\bigr) f'\bigl(x(y),r\bigr)\bigr]. \tag{6}$$

Finally, let $R(c)$ denote the Arrow–Pratt measure of relative risk aversion defined by

$$R(c) = -\frac{cu''(c)}{u'(c)} \quad \text{for all } c > 0. \tag{7}$$

We state a useful lemma that provides an estimate of the marginal rate of substitution using the Arrow–Pratt measure of relative risk aversion.

Lemma 4. *For any* $c > 0$, $\eta > 1$, *let* $\underline{R}(c,\eta) = \inf\{R(z)\colon z \in [c, \eta c]\}$, $\overline{R}(c,\eta) = \sup\{R(z)\colon z \in [c, \eta c]\}$. *Then*:

$$\frac{u'(\eta c)}{u'(c)} \leqslant \left(\frac{1}{\eta}\right)^{\underline{R}(c,\eta)}, \tag{8}$$

$$\frac{u'(\eta c)}{u'(c)} \geqslant \left(\frac{1}{\eta}\right)^{\overline{R}(c,\eta)}. \tag{9}$$

Proofs of Lemmas 1, 2 and 3 are standard in the literature and hence omitted. The proof of Lemma 4 is contained in [16].

3. The economy may not be bounded away from zero

The central focus of this paper is the possibility that consumption and capital may not be bounded away from zero in the long run even when the marginal return on investment at zero is "sufficiently large". In this section, we examine the extent of this problem and shed some light on the economic factors that lead to this problem.

3.1. The problem

To begin, consider the *deterministic* version of the stochastic growth model outlined in the previous section; in particular, suppose that the probability distribution of $f(x,r)$ is degenerate and

$$\overline{f}(x) = \underline{f}(x) = h(x), \quad \forall x \geqslant 0.$$

This is the well-known Cass–Koopmans [5,8] neoclassical optimal growth model. As is well known, if

$$\lim_{x \to 0} h'(x) > \frac{1}{\delta}, \tag{10}$$

i.e., the net marginal productivity at zero exceeds the discount rate, the sequence of optimal capital stocks from every strictly positive initial stock converges monotonically to a strictly positive limit viz., a "modified golden rule" stock x^* that satisfies $h'(x^*) = \frac{1}{\delta}$. In other words, under (10), optimal capital and consumption are always bounded away from zero and indeed, if the initial stock is small enough, capital and consumption grow over time. Further, if $\lim_{x\to 0} h'(x) = +\infty$, then (10) is satisfied for every $\delta \in (0,1)$ so that positive consumption and capital are sustained in the long run no matter how heavily the future is discounted.

The situation may, however, be qualitatively different in the stochastic version of the model. This was first pointed out by Mirman and Zilcha [11] in a striking example that we briefly summarize now. Consider the stochastic growth model outlined in the previous section with the following specific form of the production function:

$$f(x,r) = r x^{\frac{1}{2}} \tag{11}$$

and assume that the distribution F of the random shocks is the uniform distribution on the interval $[\alpha, \beta]$, $0 < \alpha < \beta$. Note that for the above production function, the (limiting) marginal productivity at zero is infinite for all possible realizations of the random shock, i.e.,

$D_+ f(0,r) = +\infty$, $\forall r \in [\alpha, \beta]$. Mirman and Zilcha show that for each $\delta \in (0,1)$, there exists a smooth "well-behaved" utility function u_δ satisfying, for instance, the assumptions on u outlined in the previous section, such that the optimal investment policy function for the economy (u_δ, f, δ, F) denoted by $x_\delta(y)$ satisfies:

$$\underline{f}\big(x_\delta(y)\big) < y, \quad \forall y > 0. \tag{12}$$

The property (12) of the policy function implies that *no matter what the current state of the economy*, capital, output and consumption necessarily decline under the worst realization of the current technology. This implies that capital and output are not bounded away from zero in the long run, and there is no invariant distribution whose support is bounded away from zero.

This example illustrates a fundamental difference between the stochastic and the deterministic growth models. For the production function (11), if the distribution of r_t is degenerate, for instance if $\alpha = \beta = 1$, the optimal capital path from every $y_0 > 0$ converges to the unique modified golden rule capital stock $x^* = \frac{\delta^2}{4} > 0$, independent of the choice of utility function u. However, even with a little bit of uncertainty in the production function, and no matter how mildly one discounts the future, there is some utility function for which capital and consumption may be driven arbitrarily close to zero in the long run.[8]

Now, in an economy with the same production function (11) as used in this example, if the utility function is given by $u(c) = \ln c$, then it is easy to check (see, for instance, [10]) that for every $\delta \in (0,1)$, the optimal policy is one where the economy expands near zero even under the worst realization of the production shock. The nature of the utility function therefore plays an important role in determining the long run destiny of the economy and in particular, the possibility of consumption and capital being bounded away from zero.

3.2. *Nowhere bounded away from zero*

In what follows, we will refer to an economy where the optimal *policy* function is of the kind described in the Mirman–Zilcha example as being *nowhere bounded away from zero*. More precisely, define the lowest optimal transition function $\underline{H}(y)$ by

$$\underline{H}(y) = \underline{f}\big(x(y)\big), \quad y \geqslant 0.$$

Thus, $\underline{H}(y)$ is the lower bound of the support of output *next period* when the current output is y and the optimal investment policy function $x(y)$ is used to determine the amount invested.

Definition 1. The economy (u, δ, f, F) is *nowhere bounded away from zero* (*NBZ*) if the optimal policy satisfies:

$$\underline{H}(y) < y, \quad \forall y > 0. \tag{13}$$

Using Proposition 2 in [14], one can check that (13) implies that for every initial stock $y_0 = y > 0$, the stochastic process of optimal output $\{y_t(y,\omega)\}$ defined by (5) must satisfy the property:

$$\Pr\Big\{\lim_{t\to\infty} \inf y_t(y,\omega) = 0\Big\} = 1$$

[8] In a subsequent paper, Mirman and Zilcha [12] show that the same is true even if $\delta = 1$.

so that output and capital are arbitrarily close to zero infinitely often with probability one; in particular, the Markov process $\{y_t(y,\omega)\}$ has no invariant distribution whose support is bounded away from zero.

Mirman and Zilcha use the example described above to show that given a specific form of f and F, for each $\delta \in (0,1)$, there is some u (that potentially depends on δ) for which the economy exhibits NBZ. However, they do not explicitly specify the utility function nor does their analysis shed light on the qualitative properties of the utility functions for which the economy exhibits NBZ (given the stochastic technology). This, in particular, makes it difficult to understand the economics behind the phenomenon, the extent of the problem and to derive sufficiently tight conditions under which the problem does not arise. In the rest of this section, we will attempt to address this issue.

Consider the production function:

$$f(x,r) = rx^{\gamma} \quad \text{for all } x \in \mathbb{R}_+ \text{ and } r \in A \tag{14}$$

where $A = [\alpha, \beta]$, with $0 < \alpha < \beta < \infty$ and $\gamma \in (0,1)$. The common distribution F is the uniform distribution function given by

$$F(r) = \begin{cases} 0 & \text{for } r < \alpha, \\ (r-\alpha)/(\beta-\alpha) & \text{for } \alpha \leqslant r \leqslant \beta, \\ 1 & \text{for } r > \beta. \end{cases} \tag{15}$$

Note that the production function considered in the Mirman–Zilcha example is a special case of this where $\gamma = \frac{1}{2}$. Also, observe that $D_+ f(0,r) = +\infty$, for all $r \in [\alpha, \beta]$. We begin with a sufficient condition for NBZ when the discount factor is small enough.

Let

$$\lambda = \frac{\beta}{\alpha}.$$

Clearly, $\lambda > 1$.

Proposition 1. *Consider the stochastic technology* (f, F) *described by* (14) *and* (15). *Suppose that*:

$$\limsup_{c \to 0} \int_1^{\lambda} \left[\frac{1}{c^{\frac{1}{\gamma}-1}} \left\{ \frac{u'(\mu^{\theta} c)}{u'(c)} \right\} \right] d\mu < \infty \tag{16}$$

for some $\theta \in (0,1)$. *Then, there exists* $\delta_0 > 0$ *such that for every* $\delta \in (0, \delta_0)$, *the economy* (f, F, u, δ) *is nowhere bounded away from zero.*

Proof. Suppose not. Choose any $\delta \in (0,1)$ for which the economy (f, F, u, δ) does not exhibit NBZ; i.e., there exists $y > 0$ such that $\underline{f}(x(y)) = \alpha(x(y))^{\gamma} \geqslant y$. Using (T.3), (T.5) and the continuity of $x(y)$, there exists $z > 0$ such that

$$\alpha x(z)^{\gamma} = z, \qquad \alpha x(y)^{\gamma} < y \quad \text{for all } y > z. \tag{17}$$

We will show that if u satisfies (16), then there exists $\delta_0 > 0$ such that for $\delta < \delta_0$, (17) leads to a contradiction. This is accomplished in three steps:

Step 1. We use the stochastic Ramsey–Euler equation to derive the following inequality:

$$\frac{\delta\lambda\gamma}{\lambda-1}\alpha^{\frac{1}{\gamma}}\left(\frac{1}{z}\right)^{\frac{1-\gamma}{\gamma}}\int_1^\lambda \frac{u'(c(\mu z))}{u'(c(z))}\,d\mu > 1. \tag{18}$$

Step 2. We show that for δ small enough,

$$c(\mu z) > \mu^\theta c(z). \tag{19}$$

Step 3. Using (19) in the inequality (18), we obtain a lower bound on δ which (using (16)) remains bounded away from zero even as $c \to 0$, so that there is a contradiction if δ is close enough to zero.

Step 1. For the specific production function and distribution of shocks, the Ramsey–Euler equation (6) can be written as:

$$u'\big(c(y)\big) = \delta\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta u'\big(c\big(rx(y)^\gamma\big)\big)\gamma r x(y)^{\gamma-1}\,dr\right] \quad \text{for all } y>0.$$

As $r<\beta$ with positive probability and $r \leqslant \beta$ with probability one, we have for all $y>0$,

$$\frac{\delta\beta\gamma}{\beta-\alpha}x(y)^{\gamma-1}\int_\alpha^\beta \frac{u'(c(rx(y)^\gamma))}{u'(c(y))}\,dr > 1$$

and letting $\lambda = \frac{\beta}{\alpha}$, the change of variable $\mu = \frac{r}{\alpha}$ yields

$$\frac{\delta\lambda\gamma}{\lambda-1}\alpha x(y)^{\gamma-1}\int_1^\lambda \frac{u'(c(\alpha\mu x(y)^\gamma))}{u'(c(y))}\,d\mu > 1 \quad \text{for all } y>0. \tag{20}$$

From (17), $\alpha x(z)^\gamma = z$ so that $x(z)^{\gamma-1} = (\frac{\alpha}{z})^{\frac{1-\gamma}{\gamma}}$ and using this in (20), we obtain (18).

Step 2. Using Lemma 2, $c(\mu z) \geqslant c(z)$ and the concavity of u, we have

$$\int_1^\lambda \frac{u'(c(\mu z))}{u'(c(z))}\,d\mu \leqslant \lambda - 1$$

and using this in (18) yields:

$$\delta\lambda\gamma > \frac{1}{z}\left(\frac{z}{\alpha}\right)^{\frac{1}{\gamma}}. \tag{21}$$

From (17), $\alpha x(z)^\gamma = z$ and $\alpha x(\mu z)^\gamma < \mu z$ for $\mu > 1$. Therefore,

$$c(z) = z - x(z) = z - \left(\frac{z}{\alpha}\right)^{\frac{1}{\gamma}}, \tag{22}$$

$$c(\mu z) = \mu z - x(\mu z) > \mu z - \left(\frac{\mu z}{\alpha}\right)^{\frac{1}{\gamma}}. \tag{23}$$

We now claim that there exists $\delta_1 > 0$ such that

$$\mu z - \left(\frac{\mu z}{\alpha}\right)^{\frac{1}{\gamma}} > \mu^{\theta}\left(z - \left(\frac{z}{\alpha}\right)^{\frac{1}{\gamma}}\right) \quad \text{for all } \delta \in (0, \delta_1),\ \mu \in (1, \lambda). \tag{24}$$

(Note that in the above inequality, z depends on δ.) To establish (24), note that $\frac{\mu^{1-\theta}-1}{\mu^{\frac{1}{\gamma}-\theta}-1} > 0$ for all $\mu > 1$ and (using L'Hôpital's rule)

$$\lim_{\mu \downarrow 1} \frac{\mu^{1-\theta} - 1}{\mu^{\frac{1}{\gamma}-\theta} - 1} = \frac{1-\theta}{\frac{1}{\gamma} - \theta} > 0$$

so that

$$\inf_{\mu \in (1,\lambda]} \frac{\mu^{1-\theta} - 1}{\mu^{\frac{1}{\gamma}-\theta} - 1} > 0.$$

Therefore, there exists $\delta_1 > 0$ such that for $\delta \in (0, \delta_1)$ and all $\mu \in (1, \lambda)$, $\frac{\mu^{1-\theta}-1}{\mu^{\frac{1}{\gamma}-\theta}-1} > \delta\gamma\lambda$ which, when combined with (21), yields

$$\frac{\mu^{1-\theta} - 1}{\mu^{\frac{1}{\gamma}-\theta} - 1} > \frac{1}{z}\left(\frac{z}{\alpha}\right)^{\frac{1}{\gamma}}.$$

Multiplying through by $z(\mu^{\frac{1}{\gamma}-\theta} - 1)$ yields (24). In what follows, let $\delta \in (0, \delta_1)$. Using (24), (23) and (22), we have that for all $\mu \in (1, \lambda)$,

$$c(\mu z) > \mu z - \left(\frac{\mu z}{\alpha}\right)^{\frac{1}{\gamma}} > \mu^{\theta}\left(z - \left(\frac{z}{\alpha}\right)^{\frac{1}{\gamma}}\right) = \mu^{\theta} c(z). \tag{25}$$

Thus, (19) holds.

Step 3. Using (18) and (19) we have:

$$1 < \frac{\delta\lambda\gamma}{\lambda - 1}\alpha^{\frac{1}{\gamma}}\left(\frac{1}{z}\right)^{\frac{1-\gamma}{\gamma}} \int_1^{\lambda} \frac{u'(c(\mu z))}{u'(c(z))}\, d\mu < \frac{\delta\lambda\gamma}{\lambda - 1}\alpha^{\frac{1}{\gamma}} \int_1^{\lambda} \frac{u'(\mu^{\theta} c(z))}{z^{\frac{1-\gamma}{\gamma}} u'(c(z))}\, d\mu$$

$$< \frac{\delta\lambda\gamma}{\lambda - 1}\alpha^{\frac{1}{\gamma}} \int_1^{\lambda} \frac{u'(\mu^{\theta} c(z))}{c(z)^{\frac{1-\gamma}{\gamma}} u'(c(z))}\, d\mu, \quad \text{as } c(z) < z,$$

i.e.,

$$\delta > \frac{\lambda - 1}{\lambda\gamma\alpha^{\frac{1}{\gamma}}}\left(\int_1^{\lambda} \frac{u'(\mu^{\theta} c)}{c^{\frac{1-\gamma}{\gamma}} u'(c)}\, d\mu\right)^{-1} \tag{26}$$

where $c = c(z) \in (0, K]$. However, using (16), there exists $M > 0$ such that

$$\int_1^{\lambda} \frac{u'(\mu^{\theta} c)}{c^{\frac{1-\gamma}{\gamma}} u'(c)}\, d\mu < M \quad \text{for all } c \in (0, K]$$

and therefore, (26) implies $\delta > \frac{\lambda-1}{\lambda\gamma\alpha^{\frac{1}{\gamma}}}\frac{1}{M}$. Setting $\delta_0 = \min\{\delta_1, \frac{\lambda-1}{\lambda\gamma\alpha^{\frac{1}{\gamma}}}\frac{1}{M}\} > 0$, we arrive at a contradiction for $\delta \in (0, \delta_0)$. The proof is complete. □

For the specific stochastic technology, Proposition 1 provides a *verifiable* sufficient condition (16) on the utility function for the economy to be nowhere bounded away from zero when the future is discounted sufficiently. The integrand on the left-hand side of the inequality in (16) consists of two terms. As $\gamma \in (0, 1)$, $c^{\frac{1}{\gamma}-1} \to 0$ as $c \to 0$, and the rate at which this occurs depends on γ. On the other hand, the term $\frac{u'(\mu^\theta c)}{u'(c)} < 1$ is the marginal rate of substitution between consumption in the current and next time periods when the consumption grows by a factor μ^θ, where $\mu^\theta = (\frac{r}{\alpha})^\theta$ is a lower bound on the rate of consumption growth at the largest positive fixed point, if any, of $\underline{H}(y)$. The condition (16) requires that this marginal rate of substitution converge to zero as $c \to 0$ at a rate that outweighs the rate at which $c^{\frac{1}{\gamma}-1} \to 0$. Observe that the ratio $\frac{u'(\mu^\theta c)}{u'(c)}$ depends on the curvature of the utility function and therefore, the degree of risk aversion. The higher the degree of risk aversion, the smaller the ratio $\frac{u'(\mu^\theta c)}{u'(c)}$. In Lemma 4, we provide explicit bounds on this ratio in terms of the Arrow–Pratt measure of relative risk aversion; if this measure diverges to ∞ as $c \to 0$, then the ratio $\frac{u'(\mu^\theta c)}{u'(c)} \to 0$ and the rate at which this occurs depends on the rate at which risk aversion becomes infinitely large at zero.

In what follows, we derive an explicit necessary and sufficient condition on the utility function for the economy to be nowhere bounded away from zero with sufficient discounting. In order to obtain a simple condition that is easy to understand and verify, we impose a further restriction. The utility function is assumed to satisfy decreasing relative risk aversion in a neighborhood of zero

$$\exists s > 0 \quad \text{such that} \quad R(c) \text{ is non-increasing in } c \text{ on } (0, s). \tag{27}$$

As we will see in Proposition 5 of the next section, if $R(c)$ is bounded as $c \to 0$, then infinite expected marginal productivity at zero (as exhibited by the production function (14)) implies that the economy is always bounded away from zero. Therefore, for the economy to exhibit NBZ, it is necessary that $R(c) \to \infty$ as $c \to 0$. Condition (27) is therefore not a very strong restriction.

Proposition 2. *Consider the stochastic technology* (f, F) *described by* (14) *and* (15). *Suppose that the utility function u satisfies* (27). *If*

$$\liminf_{c\to 0}\left[R(c)c^{\frac{1}{\gamma}-1}\right] > 0, \tag{28}$$

then there exists $\delta_0 > 0$ *such that for every* $\delta \in (0, \delta_0)$, *the economy* (u, δ, f, F) *is nowhere bounded away from zero. Conversely, if there is some* $\delta \in (0, 1)$ *such that the economy* (u, δ, f, F) *is nowhere bounded away from zero, then* (28) *holds.*

Proof. First, we show that if (28) holds then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, the economy (u, δ, f, F) is nowhere bounded away from zero. Let $\theta \in (0, 1)$ be given. In view of Proposition 1, it suffices to show that:

$$\limsup_{c\to 0}\left[\frac{1}{c^{\frac{1}{\gamma}-1}}\int_1^\lambda \frac{u'(\mu^\theta c)}{u'(c)}\,d\mu\right] < \infty. \tag{29}$$

Note that since $R(c)$ is decreasing in c on $(0, s)$, condition (28) implies:

$$R(c) \uparrow \infty \quad \text{as } c \downarrow 0 \text{ on } (0, s). \tag{30}$$

In particular, there exists $c' \in (0, \frac{s}{\lambda^\theta})$ such that

$$R\big(\lambda^\theta c\big) > \frac{1}{\theta} \quad \text{for all } c \in \big(0, c'\big). \tag{31}$$

Choose an arbitrary $c \in (0, c')$. Using Lemma 4 and the fact that for $c \in (0, c')$, $R(.)$ is non-increasing on $[c, \lambda^\theta c]$, we have that for all $\mu \in [1, \lambda]$:

$$\left[\frac{u'(\mu^\theta c)}{u'(c)}\right] \leqslant \frac{1}{\mu^{\theta R(\mu^\theta c)}} \leqslant \frac{1}{\mu^{\theta R(\lambda^\theta c)}} \tag{32}$$

and so:

$$\begin{aligned}\int_1^\lambda \left[\frac{u'(\mu^\theta c)}{u'(c)}\right] d\mu &\leqslant \int_1^\lambda \left[\frac{1}{\mu^{\theta R(\lambda^\theta c)}}\right] d\mu \\ &= \int_1^{\lambda^\theta} \left[\frac{1}{\theta t^{R(\lambda^\theta c) - \frac{1}{\theta} + 1}}\right] dt, \quad \text{after a change of variable } t = \mu^\theta \\ &= \left[\frac{1 - \lambda^{1-\theta R(\lambda^\theta c)}}{\theta R(\lambda^\theta c) - 1}\right] \leqslant \frac{1}{\theta R(\lambda^\theta c) - 1}. \end{aligned} \tag{33}$$

Using (33),

$$\begin{aligned}\frac{1}{c^{\frac{1}{\gamma}-1}} \int_1^\lambda \frac{u'(\mu^\theta c)}{u'(c)} d\mu &\leqslant \frac{1}{[\theta R(\lambda^\theta c) - 1] c^{\frac{1}{\gamma}-1}} \\ &= \frac{\lambda^{\theta(\frac{1}{\gamma}-1)}}{\theta[R(\lambda^\theta c)(\lambda^\theta c)^{\frac{1}{\gamma}-1}] - (\lambda^\theta c)^{\frac{1}{\gamma}-1}}. \end{aligned} \tag{34}$$

From (28), we have that $\lim_{c\to 0} \inf[R(\lambda^\theta c)(\lambda^\theta c)^{\frac{1}{\gamma}-1}] > 0$, and since $(\lambda^\theta c)^{\frac{1}{\gamma}-1} \to 0$ as $c \to 0$, (34) implies that (29) holds. This completes the proof of the first part of the proposition.

Next, we show that if there is some $\delta \in (0, 1)$ such that the economy (u, δ, f, F) is nowhere bounded away from zero, then (28) holds. Suppose to the contrary that there exists $\delta \in (0, 1)$ such that the economy (u, δ, f, F) is nowhere bounded away from zero, but:

$$\lim_{c\to 0} \inf\big[R(c) c^{(\frac{1}{\gamma}-1)}\big] = 0. \tag{35}$$

Proposition 8 in Section 5 provides a sufficient condition (80) for the economy to be bounded away from zero (BAZ), a property that violates NBZ. Since the economy (u, δ, f, F) exhibits NBZ, (80) cannot hold and therefore:

$$\delta \frac{\gamma}{\beta - \alpha} \left[\limsup_{x\to 0} \int_\alpha^\beta \frac{u'(rx^\gamma)}{u'(\alpha x^\gamma - x) x^{1-\gamma}} r\, dr\right] \leqslant 1$$

and setting $\mu = \frac{r}{\alpha}$, $\lambda = \frac{\beta}{\alpha}$, we have:

$$\lim_{x\to 0}\sup \int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\alpha x^{\gamma}-x)}\right] d\mu < \infty. \tag{36}$$

We will establish a contradiction to (36). The integrand on the left-hand side of (36) can be written as:

$$\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\alpha x^{\gamma}-x)} = \frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\mu\alpha x^{\gamma})}\big[J(x)\big]^{-1} \tag{37}$$

where $J(x) = \frac{u'(\alpha x^{\gamma}-x)}{u'(\alpha x^{\gamma})}$. Later, we will show that there exists $x' > 0$ such that

$$J(x) \leqslant 2 \quad \text{for all } x \in \big(0, x'\big). \tag{38}$$

Using (38) in (37) we have for $x \in (0, x')$:

$$\int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\alpha x^{\gamma}-x)}\right] d\mu \geqslant \frac{1}{2}\int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\alpha x^{\gamma})}\right] d\mu. \tag{39}$$

Since the economy exhibits NBZ, using Proposition 5 we have that $\lim_{c\to 0}\sup R(c) = \infty$. Further, since $R(c)$ is non-increasing in c on $(0, s]$, we have $R(c) \uparrow \infty$ as $c \downarrow 0$ on $(0, s)$. Therefore, there exists $x'' \in (0, x')$ such that $R(\alpha x^{\gamma}) > 1$ for all $x \in (0, x'')$. Choose any $x \in (0, x'')$. As $R(c)$ is non-increasing on $(0, x')$, we have from Lemma 4,

$$\frac{u'(\mu\alpha x^{\gamma})}{u'(\alpha x^{\gamma})} \geqslant \left(\frac{1}{\mu}\right)^{R(\alpha x^{\gamma})}.$$

Thus, for $x \in (0, x'')$

$$\int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\alpha x^{\gamma})}\right] d\mu \geqslant \frac{1}{x^{1-\gamma}}\int_1^{\lambda} \mu^{-R(\alpha x^{\gamma})}\, d\mu$$
$$= \frac{\alpha^{\frac{1}{\gamma}-1}(1-\lambda^{1-R(\alpha x^{\gamma})})}{(\alpha x^{\gamma})^{\frac{1}{\gamma}-1}R(\alpha x^{\gamma}) - (\alpha)^{\frac{1}{\gamma}-1}x^{1-\gamma}}. \tag{40}$$

Using (35), $(\alpha x^{\gamma})^{\frac{1}{\gamma}-1}R(\alpha x^{\gamma}) \to 0$ as $x \to 0$. Further, as $x \to 0$, $x^{1-\gamma} \to 0$ and (since $R(c) \uparrow \infty$ as $c \downarrow 0$), $\lambda^{1-R(\alpha x^{\gamma})} \to 0$. Using this in (40) we have

$$\int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma}u'(\alpha x^{\gamma})}\right] d\mu \to \infty \quad \text{as } x \to 0$$

so that (using (39)) we obtain a contradiction to (36).

It remains to show that (38) holds for some $x' > 0$. There exists $x' > 0$ such that for all $x \in (0, x')$,

$$x^{1-\gamma} \leqslant \left(\frac{\alpha}{2}\right), \beta x^{\gamma} < s. \tag{41}$$

From (41),

$$\left(\alpha x^{\gamma} - x\right) = (\alpha/2)x^{\gamma} + \left[(\alpha/2) - x^{1-\gamma}\right]x^{\gamma} \geqslant (\alpha/2)x^{\gamma}. \tag{42}$$

In addition, using (35), one can choose x' small enough so that for all $x \in (0, x')$

$$R\left(\frac{\alpha}{2}x^{\gamma}\right)\left\{\frac{\alpha}{2}x^{\gamma}\right\}^{\frac{1}{\gamma}-1}\left[\frac{2}{\alpha}\right]^{(1/\gamma)} < \frac{1}{2}. \tag{43}$$

Observe that

$$J(x) = \frac{[u'(\alpha x^{\gamma} - x) - u'(\alpha x^{\gamma})] + u'(\alpha x^{\gamma})}{u'(\alpha x^{\gamma})} = 1 + \frac{[-u''(m)]x}{u'(\alpha x^{\gamma})} \tag{44}$$

where $m \in [\alpha x^{\gamma} - x, \alpha x^{\gamma}]$ is given by the mean value theorem. Since $R(c)$ is non-increasing on $(0, s)$, $[-u''(c)]$ is decreasing in c on $(0, \beta x^{\gamma}]$, we have from (44) and (42):

$$\begin{aligned} J(x) &\leqslant 1 + \frac{[-u''(\alpha x^{\gamma} - x)]x}{u'(\alpha x^{\gamma})} \leqslant 1 + R\left(\alpha x^{\gamma} - x\right)J(x)x^{1-\gamma}(2/\alpha) \\ &\leqslant 1 + R\left((\alpha/2)x^{\gamma}\right)J(x)x^{1-\gamma}(2/\alpha) \leqslant 1 + (1/2)J(x) \end{aligned}$$

where the last inequality follows from (43). Thus, (38) holds. The proof is complete. □

Condition (28) provides a tight characterization of the kind of utility function that can lead to the phenomenon illustrated in the Mirman–Zilcha example. This condition brings out explicitly the tension between marginal productivity becoming infinitely large at zero (at a rate depending on γ) and the degree of risk aversion exploding as consumption goes to zero. It requires that as $c \to 0$, risk aversion $R(c) \to \infty$ at a rate faster than the rate at which $c^{(\frac{1}{\gamma}-1)} \to 0$.

Example 1. Consider the "expo-power" utility function[9]:

$$u(c) = -\exp\left(pc^{-q}\right), \quad \text{for } c > 0$$

where $p > 0$, $q > 0$. It is easy to check that this utility function satisfies (U.1), (U.2), (U.3) and (27). The Arrow–Pratt measure of relative risk-aversion for this utility function is given by

$$R(c) = 1 + q + pqc^{-q}$$

so that

$$R(c)c^{\frac{1}{\gamma}-1} = (1+q)c^{\frac{1}{\gamma}-1} + pqc^{-q+\frac{1}{\gamma}-1}$$

and therefore, (28) is satisfied if, and only if, $\gamma \geqslant \frac{1}{1+q}$.

[9] The exponential-power utility functions used in Example 1 and in Proposition 3 have been characterized and used in the literature on risk aversion; see, for instance, [18] and [19].

The results outlined above provide explicit and verifiable conditions on the utility function under which the economy (with the Cobb–Douglas production function and multiplicative shock) is nowhere bounded away from zero provided the future is discounted sufficiently, i.e., δ is small enough. Mirman and Zilcha showed that for *every* $\delta \in (0,1)$, there exists some utility function for which the economy is nowhere bounded away from zero. This leads us to the question about whether we can *explicitly* specify a utility function for which the economy is nowhere bounded away from zero even when discounting is sufficiently mild. The next proposition outlines such a condition.

Choose $\delta' \in (0,1)$. Consider the (uniform) distribution function F with support $[\alpha,\beta]$ as described in (15). We impose the following restriction on the parameters of the distribution:

$$\alpha = 1 < \beta < \frac{2}{(\delta')^{\frac{1}{4}}} - 1. \tag{45}$$

Clearly, if $\beta > 1$ is chosen sufficiently close to 1, then (45) can be satisfied. The production function f is given by

$$f(x,r) = r\sqrt{x}, \quad r \in [\alpha,\beta],\ x \geqslant 0. \tag{46}$$

Finally, let the utility function $u : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be given by

$$u(c) = \begin{cases} -e^{(1/c^m)} & \text{if } c > 0, \\ -\infty & \text{if } c = 0, \end{cases} \quad m > 1. \tag{47}$$

We will now impose a restriction on the parameter m, given β. For $t \in I \equiv (0,(1/\beta))$, define:

$$\phi(t) = 1 - \frac{\beta t}{(2-t)}. \tag{48}$$

Note that ϕ maps I to $\mathbb{R}_{++}$, and $\phi(t) \to 1$ as $t \to 0$. Choose $\theta \in I$ such that:

$$\left.\begin{array}{ll} \text{(i)} & \phi(\theta) > \sqrt{\delta'}, \\ \text{(ii)} & 2^{(1/\theta)} > \dfrac{\beta^2}{(\beta-1)}. \end{array}\right\} \tag{49}$$

Clearly, if θ is chosen sufficiently close to 0, then both conditions in (49) can be satisfied.[10] Fix any such θ and set

$$m = (1/\theta). \tag{50}$$

Proposition 3. *Choose any $\delta' \in (0,1)$. Given δ', consider the utility function u defined in* (47)*, the production function f defined in* (46)*, and the distribution F for the random shocks defined in* (15) *subject to parametric restrictions* (45), (48), (49) *and* (50)*. Then, for every $\delta \in (0,\delta')$, the economy (f,F,u,δ) is nowhere bounded away from zero.*

Though the mathematical details differ, the main steps of the proof of Proposition 3 closely resemble those of the proof of Proposition 1. A complete proof of Proposition 3 is contained in [16].

[10] Suppose we choose $\delta' = 0.9801$. Then, (45) can be satisfied by choosing $\beta = 1.01$. Further, by choosing $\theta = 0.01$, both conditions in (49) are satisfied.

Note that the restrictions on the parameters of the utility function u and the distribution F in Proposition 3 depend on the given δ', but not on the $\delta \in (0, \delta')$.

Proposition 3 indicates that for each value of the discount factor no matter how close to 1, we can explicitly specify a stochastic technology and a utility function for which the economy is nowhere bounded away from zero. This might give rise to the suspicion that if we *fix* the technology and the utility function, and then choose a discount factor arbitrarily close to one, the economy must be bounded away from zero. We have reasons to believe that this is probably not true (though in this paper we do not explicitly study the behavior of the economy as discounting vanishes). Mirman and Zilcha [12, Remark 3] analyze the *undiscounted* stochastic growth model and extend the example in [11] to the undiscounted case, which implies that for the production function (11) with uniformly distributed shock, there exists a utility function for which the economy exhibits NBZ even when $\delta = 1$. Indeed, unlike the deterministic growth model, in the stochastic growth model (to the best of our knowledge) there is no general "neighborhood turnpike" result that indicates that as $\delta \uparrow 1$, optimal paths of an economy (that is productive near zero) are necessarily bounded away from zero.

3.3. NBZ when productivity is bounded

The Mirman–Zilcha example focuses on an economy where marginal productivity is infinitely large near zero; the fact that even such an economy may not be bounded away from zero brings out the problem of sustaining positive consumption under uncertainty rather sharply. Our analysis indicates that the problem illustrated in the Mirman–Zilcha example arises if risk aversion diverges to infinity at a sufficiently fast rate as consumption goes to zero. The reader may infer from this that the problem does not arise if risk aversion is bounded as, for instance, in case of the constant relative risk aversion (CRRA) utility function which is widely used. This (as we show in the next section) is indeed true if the marginal productivity at zero is infinite. *If, however, the marginal productivity at zero is finite* (*which is perhaps more realistic, though often assumed away for the sake of convenience*)*, then the economy may be nowhere bounded away from zero even if risk aversion is bounded*. In other words, there is no family of utility functions that is "safe" independent of production technology. We report below the gist of an example that is developed fully in a separate paper which illustrates this point.

Example 2. (See Mitra and Roy, [15].) Suppose the utility function satisfies constant relative risk aversion (*CRRA*) with relative risk aversion parameter ρ; i.e., $u : \mathbb{R}_{++} \to \mathbb{R}$ is given by

$$u(c) = \begin{cases} \frac{c^{1-\rho}}{1-\rho} & \text{if } \rho \neq 1, \\ \ln c & \text{otherwise} \end{cases}$$

with $u(0) = \lim_{c \to 0} u(c)$ when $\rho \in (0, 1)$, and $u(0) = -\infty$ otherwise. Let

$$f(x, r) = rh(x) \quad \text{for all } x \in \mathbb{R}_+ \text{ and } r \in A$$

where $A = [1, \beta]$, with $1 < \beta < \infty$, and:

$$h(x) = Bx/(1 + x) \quad \text{for all } x \in \mathbb{R}_+$$

with $B > 1$. Let F be the uniform distribution on $[1, \beta]$. Assume $\delta \in (0, 1)$ satisfies

$$\delta(Er)h'(0) = \delta(Er)B > 1.$$

However, $\delta(Er)h'(0)$ is close enough to 1. Then, one can explicitly specify ρ' such that for all $\rho > \rho'$, $h(x(y)) < y$ for all $y > 0$ and the economy is nowhere bounded away from zero.

4. Growth with certainty near zero (GNZ)

In this section, we consider *the general version of the model* (as outlined in Section 2) and focus on a strong concept of sustaining positive consumption. The concept requires that when the current output is close enough to zero, the economy expands its capital stock even under the worst realization of the production technology. We shall refer to this as *growth with certainty near zero*.

4.1. The concept

Recall that $\underline{H}(y) = \underline{f}(x(y))$ is the lower bound of the support of output *next period* when the current output is y and the optimal investment policy function $x(y)$ is used to determine the amount invested.

Definition 2. The economy (u, δ, f, F) exhibits *growth with certainty near zero* (*GNZ*) if there exists $\alpha > 0$ such that

$$\underline{H}(y) > y, \quad \forall y \in (0, \alpha). \tag{51}$$

Consider $\{y_t(y, \omega)\}$, the Markov process of optimal output from initial stock $y > 0$, defined by (5). Let $\{c_t(y, \omega)\}$ be the Markov process of optimal consumption from initial stock $y > 0$ defined by

$$c_t(y, \omega) = c\big(y_t(y, \omega)\big).$$

(51) implies that $f(x(y), r) > y$, i.e., $y_1(y, \omega) > y_0 = y$ almost surely. Indeed, for each $y \in (0, \alpha)$, there exists $T(y) \geqslant 0$

$$\Pr\big\{y_{t+1}(y, \omega) > y_t(y, \omega),\ \forall t = 0, \dots, T(y)\big\} = 1,$$

i.e., the economy grows with probability one for at least $T(y)$ periods if the current stock is small enough. Thus, GNZ ensures sufficiently poor economies experience some growth (almost surely) on their transition path. Indeed, GNZ implies that for all $y > 0$,

$$\Pr\Big\{\lim_{t\to\infty} \inf y_t(y, \omega) > \alpha\Big\} = 1,$$

i.e., independent of initial stock y, optimal output eventually exceeds α with probability one. Using Lemma 2, we have then:

$$\Pr\Big\{\lim_{t\to\infty} \inf c_t(y, \omega) > c(\alpha)\Big\} = 1,$$

i.e., independent of initial stock, optimal consumption eventually exceeds $c(\alpha) > 0$ with probability one. Thus, GNZ ensures a uniform positive lower bound on long run consumption that is independent of the initial condition of the economy.

In their pioneering analysis of the optimal stochastic growth model, Brock and Mirman [4] impose strong conditions to ensure GNZ and use this to show the existence of a unique invariant distribution whose support is bounded away from zero. The conditions they impose are as follows: the marginal productivity at zero is infinite for all realizations of the random shock and there is a strictly positive probability mass on the "worst" realization of the technology. These conditions rule out economies where the production shock is continuously distributed and economies where productivity at zero may be finite, at least for bad realizations of the technology shock. The subsequent literature on stochastic growth theory has not developed any alternative set of conditions that ensures GNZ. In this section, we develop sufficient conditions for GNZ that can be satisfied even when the distribution of the random production shock has no mass point and when marginal productivity is bounded.

4.2. General sufficient conditions for GNZ

Recall the definition of $\overline{f}(x)$ and $\underline{f}(x)$ in (1). We begin by outlining a general sufficient conditions for growth with certainty near zero.

Proposition 4. *Suppose that*

$$\delta \lim_{x\to 0} \inf E\left(\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)\right) > 1. \tag{52}$$

Then, the economy (u, δ, f, F) exhibits growth with certainty near zero.

Proof. Suppose not. Then, there exists a sequence $\{y^n\}_{n=1}^{\infty} \to 0$, $y^n \in \mathbb{R}_{++}$ such that

$$\underline{f}\big(x\big(y^n\big)\big) \leqslant y^n, \quad \forall n. \tag{53}$$

Let $x^n = x(y^n)$. Then, using Lemma 1 and (53)

$$0 < x^n < y^n, \qquad \underline{f}\big(x^n\big) \leqslant y^n, \quad \forall n \geqslant N. \tag{54}$$

Further, $\{x^n\} \to 0$. Using (6), (54) and the concavity of u, we have for all $n \geqslant N$,

$$\begin{aligned} u'\big(\underline{f}\big(x^n\big)-x^n\big) &\geqslant u'\big(y^n - x^n\big) = u'\big(c\big(y^n\big)\big) \\ &= \delta E\big[u'\big(c\big(f\big(x^n, r\big)\big)\big) f'\big(x^n, r\big)\big] \geqslant \delta E\big[u'\big(f\big(x^n, r\big)\big) f'\big(x^n, r\big)\big] \end{aligned}$$

so that

$$\delta E\left[\frac{u'(f(x^n, r))}{u'(\underline{f}(x^n)-x^n)} f'\big(x^n, r\big)\right] \leqslant 1, \quad \forall n \geqslant N.$$

As $\{x^n\} \to 0$ and $x^n > 0$, $\forall n$, we obtain a contradiction to (52). The proof is complete. □

In the deterministic version of the model, growth near zero is ensured as long as the discounted net marginal productivity at zero exceeds 1. This is often referred to as a "delta-productivity condition". One can view condition (52) as an *expected* "welfare-modified" delta-productivity condition at zero that reflects the stochastic nature of our model. The factor $(\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)})$ in (52) is a bound on the marginal rate of substitution between consumption in current and next period for each realization r of the random shock. Note that (52) allows the marginal productivity at zero to be below the discount rate for "bad" realizations of the random shock.

Next, define the function $\mu(r)$ on A by

$$\mu(r) = \lim_{x\to 0} \sup \frac{f(x,r)}{\underline{f}(x)}. \tag{55}$$

Let

$$\lambda = \lim_{x\to 0} \sup \frac{\overline{f}(x)}{\underline{f}(x)}. \tag{56}$$

Note that under assumption (T.4), $\lambda < \infty$ and further, $1 \leqslant \mu(r) \leqslant \lambda$, $\forall r \in A$.

Recall the definition of ν in (3). For $r \in A$, define $n(r)$ by

$$\begin{aligned} n(r) &= \mu(r)\frac{\nu}{\nu - 1} \quad \text{if } \nu < +\infty \\ &= \mu(r) \quad \text{if } \nu = +\infty. \end{aligned} \tag{57}$$

Let $\bar{n}$ be defined by

$$\begin{aligned} \bar{n} &= \lambda\frac{\nu}{\nu - 1} \quad \text{if } \nu < +\infty \\ &= \lambda \quad \text{if } \nu = +\infty. \end{aligned} \tag{58}$$

Then

$$\mu(r) \leqslant n(r) \leqslant \bar{n}, \quad \forall r \in A. \tag{59}$$

Note that if the production function has the form:

$$f(x, r) = rh(x) \tag{60}$$

where the random shock is multiplicative and $\alpha = \inf A > 0$, $\beta = \sup A > \alpha$, then

$$\mu(r) = \frac{r}{\alpha}, \qquad \lambda = \frac{\beta}{\alpha}.$$

In that case, if $h'(0) = \lim_{x\to 0} h'(x)$, then

$$\begin{aligned} n(r) &= \frac{r}{\alpha}\left(\frac{\alpha h'(0)}{\alpha h'(0) - 1}\right), \qquad \bar{n} = \frac{\beta}{\alpha}\left(\frac{\alpha h'(0)}{\alpha h'(0) - 1}\right), \quad \text{if } h'(0) < \infty, \\ n(r) &= \frac{r}{\alpha}, \qquad \bar{n} = \frac{\beta}{\alpha}, \quad \text{if } h'(0) = \infty. \end{aligned}$$

The next result refines the sufficient condition in Proposition 4 and provides a more transparent condition for GNZ.

Corollary 1. *Suppose that*

$$\delta E\left[\lim_{x\to 0} \inf \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x, r)\right] > 1. \tag{61}$$

for some $\sigma > 1$. *Then, the economy* (u, δ, f, F) *exhibits growth with certainty near zero.*

Proof. Observe that

$$\lim_{x\to 0} \sup \frac{f(x, r)}{\underline{f}(x) - x} = \lim_{x\to 0} \sup \frac{f(x, r)}{\underline{f}(x)} \frac{\frac{\underline{f}(x)}{x}}{\frac{\underline{f}(x)}{x} - 1} = n(r).$$

Since $\sigma > 1$, for each $r \in A$, there exists $y(r) > 0$ such that for all $x \in (0, y(r))$, $\frac{f(x,r)}{\underline{f}(x)-x} < \sigma n(r)$ so that

$$\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r) > \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r), \quad \forall x \in \bigl(0, y(r)\bigr).$$

In particular,

$$\liminf_{x\to 0} \frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r) \geqslant \liminf_{x\to 0} \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r). \tag{62}$$

Choose any sequence $\{x_k\} \to 0$. For each k, the function $\frac{u'(f(x_k,r))}{u'(\underline{f}(x_k)-x_k)} f'(x_k,r) \geqslant 0$ and is integrable (with respect to the probability measure for the distribution F). Using Fatou's lemma[11]:

$$\begin{aligned}\delta \liminf_{k\to\infty} E\left[\frac{u'(f(x_k,r))}{u'(\underline{f}(x_k)-x_k)} f'(x_k,r)\right] &\geqslant \delta E\left[\liminf_{k\to\infty} \frac{u'(f(x_k,r))}{u'(\underline{f}(x_k)-x_k)} f'(x_k,r)\right]\\ &\geqslant \delta E\left[\liminf_{k\to\infty} \frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)\right]\end{aligned}$$

and as this holds for every sequence $\{x_k\} \to 0$ we have that

$$\begin{aligned}&\delta \liminf_{x\to 0} E\left[\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)\right]\\ &\quad\geqslant \delta E\left[\liminf_{x\to 0} \frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)\right].\\ &\quad\geqslant \delta E\left[\liminf_{x\to 0} \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r)\right] \quad \text{using (62)}\\ &\quad> 1 \quad \text{using (61).}\end{aligned}$$

Thus, (52) holds and the result follows from Proposition 4. □

Condition (61) is also an expected welfare-modified delta productivity condition at zero that has a somewhat clearer economic interpretation than (52). $(\underline{f}(x)-x)$ is the level of consumption that sustains current output $y = \underline{f}(x)$ under the worst realization of the random shock. Following investment x, it can be shown that $n(r)(\underline{f}(x)-x)$ is the maximum consumption next period for realization r of the random shock. The factor $[\frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)}]$ in (61) is a lower bound on the marginal rate of substitution between consumption in the current and next periods for realization r of the random shock. Note that this ratio is essentially of the form $\frac{u'(pc)}{u'(c)}$ where $p > 1$, and the behavior of this ratio as $c \to 0$ plays an important role in (61). The next lemma provides a useful condition for GNZ, based on the limiting behavior of this ratio.

11 See [2, Theorem 3.3].

Lemma 5. *Let* $\underline{g}(\eta) : (1, \infty) \to [0, 1]$ *be a continuous and non-increasing function such that*:

$$\liminf_{c \to 0} \frac{u'(\eta c)}{u'(c)} \geqslant \underline{g}(\eta), \quad \forall \eta > 1.$$

Suppose

$$\delta E\big[\underline{g}\big(n(r)\big) D_+ f(0, r)\big] > 1. \tag{63}$$

Then, the economy exhibits growth with certainty near zero. Suppose further that $\underline{g}(\bar{n}) > 0$. *Then, a sufficient condition for growth with certainty near zero is given by*

$$\delta E\big[D_+ f(0, r)\big] > \frac{1}{\underline{g}(\bar{n})}. \tag{64}$$

If, in particular, $E[D_+ f(0, r)] = +\infty$, *then the economy exhibits growth with certainty near zero for every* $\delta \in (0, 1)$.

Proof. Under (63) and using continuity of $\underline{g}$, there exists $\sigma > 1$ such that

$$\delta E\big[\underline{g}\big(\sigma n(r)\big) D_+ f(0, r)\big] > 1. \tag{65}$$

Since,

$$\liminf_{x \to 0} \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} \geqslant \underline{g}\big(\sigma n(r)\big)$$

for any $\epsilon > 0$, there exists $h > 0$ such that for all $x \in (0, h)$,

$$\frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} \geqslant \underline{g}\big(\sigma n(r)\big) - \epsilon,$$

so that

$$\begin{aligned} \delta E\bigg[\liminf_{x \to 0} \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x, r)\bigg] &\geqslant \delta E\Big[\big\{\underline{g}\big(\sigma n(r)\big) - \epsilon\big\} \liminf_{x \to 0} f'(x, r)\Big] \\ &= \delta E\big[\big\{\underline{g}\big(\sigma n(r)\big) - \epsilon\big\} D_+ f(0, r)\big] \end{aligned}$$

and since $\epsilon > 0$ is arbitrary

$$\begin{aligned} \delta E\bigg[\liminf_{x \to 0} \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x, r)\bigg] &\geqslant \delta E\big[\underline{g}\big(\sigma n(r)\big) D_+ f(0, r)\big] \\ &> 1 \quad \text{using (65)}. \end{aligned}$$

Thus, (61) holds and from Corollary 1, the economy exhibits growth with certainty near zero. Using the fact that $n(r) \leqslant \bar{n}$, $\forall r \in A$, and that $\underline{g}(.)$ is non-increasing, it follows that if $\underline{g}(\bar{n}) > 0$, (64) implies (63). If, in addition, $E[D_+ f(0, r)] = +\infty$, (64) is satisfied for every $\delta \in (0, 1)$. This completes the proof. □

4.3. Risk aversion and GNZ

Our discussion in Section 3 highlighted the important role played by risk aversion near zero in determining whether the economy is nowhere bounded away from zero in the long run. In this subsection, we outline sufficient conditions for growth with certainty near zero that explicitly impose restrictions on the degree of risk aversion near zero.

Recall, that $R(c)$ denotes the Arrow–Pratt measure of relative risk aversion defined by (7).

Proposition 5. *Suppose that the utility function exhibits bounded relative risk aversion so that*

$$\overline{R} \equiv \left[\lim_{c\to 0} \sup R(c)\right] < \infty. \tag{66}$$

Further, suppose that

$$\delta E\left[\left(\frac{1}{n(r)}\right)^{\overline{R}} D_+ f(0,r)\right] > 1. \tag{67}$$

Then, the economy (u, δ, f, F) *exhibits growth with certainty near zero. A sufficient condition for* (67) *is that*

$$E\big[D_+ f(0,r)\big] > \frac{(\overline{n})^{\overline{R}}}{\delta}. \tag{68}$$

If, in particular, $E[D_+ f(0,r)] = +\infty$*, then for every* $\delta \in (0,1)$*, the economy* (u, δ, f, F) *exhibits growth with certainty near zero.*

Proof. Using (59), we have from (67) that there exists $k > 1$ such that

$$\delta E\left[\left(\frac{1}{n(r)}\right)^{k\overline{R}} D_+ f(0,r)\right] > 1. \tag{69}$$

Then, we can find $\varepsilon > 0$, such that for all $c \in (0, \varepsilon)$, we have:

$$R(c) \leqslant k\overline{R}. \tag{70}$$

For any $\eta > 1$, for all $c \in (0, \frac{\varepsilon}{\eta})$, $\eta c < \varepsilon$ so that, using Lemma 4, we have

$$\frac{u'(\eta c)}{u'(c)} \geqslant \left(\frac{1}{\eta}\right)^{k\overline{R}}.$$

Defining $\underline{g}(\eta) = (\frac{1}{\eta})^{k\overline{R}}$ we can check that $\underline{g}(\eta) : (1,\infty) \to [0,1]$ is a continuous and non-increasing function such that $\lim_{c\to 0} \inf \frac{u'(\eta c)}{u'(c)} \geqslant \underline{g}(\eta)$, $\forall \eta > 1$. Using (69), we have $\delta E[\underline{g}(n(r)) \times D_+ f(0,r)] > 1$, and it follows from Lemma 5 that the economy exhibits GNZ. Using (59), we can check that (68) implies (67). The rest of the proposition follows immediately. □

Proposition 5 provides a sufficient condition for GNZ for the class of utility functions that exhibit bounded relative risk aversion. The sufficient condition (67) is an expected delta-productivity condition modified by the factor $(\frac{1}{n(r)})^{\overline{R}}$ that reflects behavior towards risk. The lower the risk aversion at zero, the easier it is for this condition to be satisfied. Further, $n(r)$ itself reflects the extent of variability in the technology, so that the condition is more easily satisfied if the extent of variability is small. Condition (68) is a more easily verifiable sufficient condition for GNZ for bounded relative risk aversion utility functions because it separates the expected delta-productivity from the factor which captures behavior towards risk, and the extent of the risk. If the expected marginal productivity at zero is infinite, then GNZ holds for all utility functions with bounded relative risk aversion. From Proposition 5 it immediately follows that:

Corollary 2. *Suppose that the utility function u exhibits constant relative risk aversion (CRRA) with $\rho > 0$ being the relative risk aversion parameter. Further, suppose that $\delta E[D_+ f(0,r)] > 1$. Let $\hat{\rho}$ be defined by*

$$\hat{\rho} = \frac{\ln \delta E[D_+ f(0,r)]}{\ln \bar{n}}.$$

Then the economy (u, δ, f, F) exhibits growth with certainty near zero if

$$\rho < \hat{\rho}. \tag{71}$$

If $E[D_+ f(0,r)] = +\infty$, then (71) is always satisfied and the economy exhibits growth with certainty near zero no matter how high the level of relative risk aversion.

One implication of the results outlined above is that if relative risk aversion is bounded and the production function exhibits infinite expected productivity at zero, then the stochastic growth model generates growth with certainty near zero independent of the level of risk aversion or indeed, of any other property of intertemporal preference; this is qualitatively similar to the behavior of the economy in the deterministic version of the model (where, independent of the utility function, infinite productivity at zero always ensures growth near zero). However, as noted in Section 3.3, if marginal productivity is bounded, then the economy may be nowhere bounded away from zero even if risk aversion is bounded and the possibility of growth with certainty near zero depends on the *level* of risk aversion. The sufficient condition (71) for GNZ when utility function exhibits CRRA reflects this role of risk aversion; in addition to $\delta E[D_+ f(0,r)] > 1$, the condition requires that relative risk aversion is less than $\hat{\rho}$; further, the smaller the discounted expected marginal productivity at zero, the more stringent this upper bound $\hat{\rho}$ on risk aversion.

4.4. GNZ when utility is bounded below

In this subsection, we focus on utility functions that are bounded below and derive specific conditions under which growth with certainty occurs near zero. Without loss of generality, we assume that

U.4. $u(0) = 0$.

We first state a useful result due to Arrow:

Lemma 6. *(See Arrow, [1].[12]) Assume* U.4. *Then,*

$$\liminf_{c \to 0} R(c) \leqslant 1.$$

[12] See Appendix [1] to Essay 3 in [1].

Using this lemma, it follows that if $R(c)$ is monotonic in a neighborhood of zero so that $\lim_{c\to 0}\sup R(c) = \lim_{c\to 0}\inf R(c)$, then under U.4, $\overline{R} \equiv [\lim_{c\to 0}\sup R(c)] \leqslant 1$ and therefore, using Proposition 5, we have the following result:

Proposition 6. *Assume* U.4. *Further, suppose that there exists* $s > 0$ *such that* $R(c)$ *is monotonic* (*non-increasing or non-decreasing*) *on* $(0, s)$. *The economy* (u, δ, f, F) *exhibits growth with certainty near zero if*

$$\delta E\left[\left(\frac{1}{n(r)}\right) D_+ f(0, r)\right] > 1. \tag{72}$$

A sufficient condition for (72) *is given by*

$$E\big[D_+ f(0, r)\big] > \frac{\overline{n}}{\delta}.$$

If, in particular, $E[D_+ f(0, r)] = +\infty$, *then for every* $\delta \in (0, 1)$ *the economy* (u, δ, f, F) *exhibits growth with certainty near zero.*

Proposition 6 provides transparent sufficient conditions for GNZ for utility functions that are bounded below; these are modified expected delta-productivity conditions that do not require exact knowledge of the degree of relative risk aversion near zero. Unfortunately, the proposition also requires that risk aversion be monotonic in a neighborhood of zero. In the rest of this section, we outline alternative conditions for GNZ that do not have such a requirement. These conditions are in terms of the first elasticity of the utility function at zero.

Let

$$\kappa = \lim_{c\to 0}\inf \frac{u'(c)c}{u(c)}, \tag{73}$$

$$K = \lim_{c\to 0}\sup \frac{u'(c)c}{u(c)}. \tag{74}$$

Then, $\kappa, K \in [0, 1]$. We begin by establishing a set of weak inequalities.

Lemma 7. *Assume* U.4. *Fix* $\eta > 1$. *Then,*

$$\lim_{c\to 0}\sup \frac{u'(\eta c)}{u'(c)} \geqslant \eta^{\kappa-1}, \tag{75}$$

$$\lim_{c\to 0}\inf \frac{u'(\eta c)}{u'(c)} \geqslant \frac{\kappa}{K}\eta^{\kappa-1}, \quad \textit{if } K > 0. \tag{76}$$

The proof of Lemma 7 is contained in [16]. The next proposition outlines a sufficient condition for GNZ that uses the first elasticity of the utility function.

Proposition 7. *Assume* U.4 *and that* $K > 0$. *Suppose that*

$$\delta \frac{\kappa}{K} E\big[\big(n(r)\big)^{\kappa-1} D_+ f(0, r)\big] > 1, \tag{77}$$

then the economy (u, δ, f, F) *exhibits growth with certainty near zero. A sufficient condition for* (77) *is*

$$E\big[D_+ f(0,r)\big] > \left[\frac{K}{\kappa}(\bar{n})^{1-\kappa}\right]\frac{1}{\delta}. \tag{78}$$

If $E[D_+ f(0,r)] = +\infty$*, then for every* $\delta \in (0,1)$ *the economy* (u, δ, f, F) *exhibits growth with certainty near zero.*

Proof. Defining

$$\underline{g}(\eta) = \frac{\kappa}{K}\eta^{\kappa-1},$$

we can check that $\underline{g}(\eta) : (1,\infty) \to [0,1]$ is a continuous and non-increasing function where (using Lemma 7):

$$\lim_{c\to 0}\inf \frac{u'(\eta c)}{u'(c)} \geqslant \underline{g}(\eta), \quad \forall \eta > 1.$$

Under (77), $\kappa > 0$ and $\delta E[\underline{g}(n(r))D_+ f(0,r)] > 1$. From Lemma 5, we have that the economy exhibits GNZ. The rest of the proposition follows immediately. □

The sufficient condition (77) for GNZ in the above proposition is, once again, a modified expected delta-productivity condition at zero. As $n(r) > 1$, the higher the first elasticity of the utility function at zero, the more likely that this condition is satisfied. Condition (78) is a more easily verifiable sufficient condition for GNZ (for utility functions that are bounded below). As in (68) above, condition (78) separates the expected delta-productivity from the factor which depends on the extent of risk, and the elasticity of the utility function (which is related to the behavior towards risk). More generally, if the *expected* marginal productivity at zero is infinite, then GNZ occurs for all utility functions that are bounded below as long as their first elasticity is bounded away from zero or alternatively, relative risk aversion is monotonic near zero.

5. Bounded away from zero (BAZ)

In the previous section, we focused on the concept of growth with certainty near zero under which the economy expands even under the worst realization of the technology when current output is sufficiently close to zero. As mentioned earlier, this implies almost sure uniform positive lower bounds for long run capital and consumption independent of initial stock. While this is certainly sufficient to ensure that from every initial stock, capital and consumption are almost surely bounded away from zero, it is by no means necessary. In this section, we discuss a weaker concept under which capital and consumption are bounded away from zero from every positive initial stock, though the lower bound may depend on the initial condition.

For $y > 0$, recall that $\{y_t(y,\omega)\}$ is the Markov process of optimal output defined by (5).

Definition 3. We say that the economy is *bounded away from zero* (*BAZ*) if for every $y > 0$, there exists $a(y) > 0$ such that

$$\Pr\Big\{\lim_{t\to\infty}\inf y_t(y,\omega) \geqslant a(y)\Big\} = 1. \tag{79}$$

While the above definition of an economy being bounded away from zero (BAZ) is in terms of the asymptotic behavior of the stochastic process of stocks generated by the optimal policy, it is easier to visualize the nature of the optimal policy *function* that generates a stochastic process

$\{y_t(y,\omega)\}$ that satisfies the above definition. To this end, consider the lowest optimal transition function $\underline{H}(y) = \underline{f}(x(y))$, $y \geqslant 0$.

Lemma 8. *Suppose there exists a sequence* $\{y^n\}_{n=1}^{\infty} \to 0$, $y^n \in \mathbb{R}_{++}$ *such that*

$$\underline{H}(y^n) \geqslant y^n, \quad \forall n.$$

Then, the economy is bounded away from zero.

Proof. Choose any initial stock $y > 0$. There exists $N \geqslant 1$, such that $y^N < y$. Set $a(y) = y^N$. Let $\{\tilde{y}_t\}_{t=0}^{\infty}$ be the deterministic sequence defined by: $\tilde{y}_0 = y$, $\tilde{y}_{t+1} = \underline{H}(\tilde{y}_t)$, $t \geqslant 0$. Then, $\Pr\{y_t(y,\omega) \geqslant \tilde{y}_t,\ \forall t \geqslant 0\} = 1$. Further, as $\underline{f}$, $x(.)$ are non-decreasing in y, $\underline{H}(y)$ is non-decreasing in y. One can check by induction, that $\tilde{y}_{t+1} = \underline{H}(\tilde{y}_t) \geqslant \underline{f}(x(a(y))) \geqslant a(y)$, $\forall t \geqslant 0$. This concludes the proof. □

Lemma 8 indicates that BAZ allows for the possibility that $\underline{H}(y)$ has an infinite number of strictly positive fixed points that converge to zero. Further, from Lemma 8, it follows immediately, that growth with certainty near zero (GNZ) implies BAZ. Therefore, the sufficient conditions for GNZ provided in the previous section are also sufficient conditions for BAZ. However, BAZ may also be ensured under slightly weaker conditions. Note that BAZ implies that if $\{c_t(y,\omega)\}$ is the Markov process of optimal consumption from initial stock $y > 0$, then

$$\Pr\Big\{\lim_{t\to\infty}\inf c_t(y,\omega) \geqslant c\big(a(y)\big)\Big\} = 1,$$

where $c(a(y))$, the optimal consumption from stock $a(y)$, is strictly positive but may depend on the initial stock y.

Chatterjee and Shukayev [6] provide sufficient conditions for BAZ (that do not necessarily ensure GNZ). Their sufficient condition requires that (i) the utility function is bounded below and (ii) $D_+ f(0,r) > \frac{1}{\delta}$, $\forall r \in A$. Condition (i) is obviously a significant restriction as it does not allow for some of the widely used utility functions in the macro growth literature where $u(0) = -\infty$. Further, as we have shown in the previous section, if the technology is sufficiently productive at zero, it is certainly possible to ensure GNZ (which is stronger than BAZ) even when $u(0) = -\infty$ if, for instance, risk aversion is bounded. Condition (ii) is also a strong restriction in that it requires the technology to be delta-productive at zero even under the worst realization of the production shock. As shown in the previous section, it is possible to ensure GNZ (and therefore, BAZ) under conditions that require the *expected* marginal productivity at zero to be large enough even if the productivity at zero is small under the worst realization of the shock.

We begin by providing a general sufficient condition for the economy to be bounded away from zero that allows for utility functions that are unbounded below.

Proposition 8. *Suppose that*

$$\delta\left[\lim_{x\to 0}\sup E\left(\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)\right)\right] > 1. \tag{80}$$

Then the economy (u,δ,f,F) *is bounded away from zero.*

Proof. Suppose not. Then, there exists $\epsilon_1 > 0$ such that

$$\underline{f}(x(y)) < y, \quad \forall y \in (0, \epsilon_1). \tag{81}$$

Using (6), and the concavity of u, for every $y \in (0, \epsilon_2)$, $u'(c(y)) \geqslant \delta E[u'(f(x(y), r)) f'(x(y), r)]$ and since, using (81), $c(y) = y - x(y) > \underline{f}(x(y)) - x(y) > 0$, $\forall y \in (0, \epsilon_2)$, we have

$$u'(\underline{f}(x(y)) - x(y)) \geqslant \delta E[u'(f(x(y), r)) f'(x(y), r)], \quad \forall y \in (0, \epsilon_2).$$

As $x(y) > 0$ and continuous in y, this implies that

$$u'(\underline{f}(x) - x) \geqslant \delta E[u'(f(x, r)) f'(x, r)], \quad \forall x \in (0, x(\epsilon_2)),$$

which contradicts (80). The proof is complete. □

This proposition can be used to derive a more transparent sufficient condition. Recall the definition of $\mu(r)$, $n(r)$ in (55) and (57).

Corollary 3. *Assume that as $x \to 0$*

$$\frac{f(x, r)}{\underline{f}(x)} \to \mu(r) \quad \textit{uniformly in } r \textit{ on } A. \tag{82}$$

If

$$\delta \limsup_{x \to 0} E\left[\frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x, r)\right] > 1, \tag{83}$$

for some $\sigma > 1$, then the economy (u, δ, f, F) is bounded away from zero.

Proof. Since $\sigma > 1$ and $\frac{f(x,r)}{\underline{f}(x)} \to \mu(r)$ uniformly in r on A as $x \to 0$, there exists $\epsilon > 0$, such that $\forall x \in (0, \epsilon)$, $\forall r \in A$,

$$\frac{f(x, r)}{\underline{f}(x) - x} = \frac{f(x, r)}{\underline{f}(x)} \frac{\frac{\underline{f}(x)}{x}}{\frac{\underline{f}(x)}{x} - 1} < \sigma n(r),$$

so that

$$\frac{u'(f(x, r))}{u'(\underline{f}(x) - x)} f'(x, r) > \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x, r).$$

Taking the lim sup as $x \to 0$ on both sides of the above inequality and using (83), condition (80) holds. The result now follows from Proposition 8. □

Note that (82) is always satisfied for the multiplicative shock production function described in (60). The sufficient condition (83) for BAZ in the above result is a weaker version of the sufficient condition (61) for GNZ in Proposition 5.

Our final proposition outlines a more easily verifiable condition for the economy to be bounded away from zero *when the utility function is bounded below*. Recall the definition of κ in (73).

Proposition 9. *Assume* U.4 *and that as* $x \to 0$

$$\frac{f(x,r)}{\underline{f}(x)} \to \mu(r) \quad \textit{uniformly in } r \textit{ on } A.$$

Suppose that

$$\delta E\big[D_+ f(0,r)\big] > (\overline{n})^{1-\kappa}. \tag{84}$$

Then, the economy (u, δ, f, F) *is bounded away from zero. In particular, if* $E[D_+ f(0,r)] = +\infty$*, then for every* $\delta \in (0,1)$*, the economy* (u, δ, f, F) *is bounded away from zero.*

Proof. Under (84), there exists $\sigma > 1$ such that $\delta(\sigma\overline{n})^{\kappa-1} E[D_+ f(0,r)] > 1$. Then,

$$\begin{aligned}
&\delta \limsup_{x\to 0} E\left[\frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r)\right] \\
&\quad \geqslant \delta \limsup_{x\to 0}\left[\frac{u'(\sigma\overline{n}(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} Ef'(x,r)\right] \\
&\quad \geqslant \delta(\sigma\overline{n})^{\kappa-1} \limsup_{x\to 0} E\big[f'(x,r)\big] \quad \text{using (75)} \\
&\quad = \delta(\sigma\overline{n})^{\kappa-1} E\big[D_+ f(0,r)\big] > 1.
\end{aligned}$$

From Corollary 3, the economy is bounded away from zero. □

The sufficient conditions for BAZ in Proposition 9 are weaker versions of and comparable to the conditions for GNZ outlined in Proposition 7. However, unlike the latter, Proposition 9 also requires that $\frac{f(x,r)}{\underline{f}(x)}$ converges uniformly in r as $x \to 0$.

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